14.7 Max/Min for surfaces

Consider the surface $z = f(x,y)$.

Some Terminology:

A **critical point** is a point (a,b) where **BOTH**

$$f_x(a,b) = 0 \quad \text{AND} \quad f_y(a,b) = 0$$

or where either partial doesn't exist.

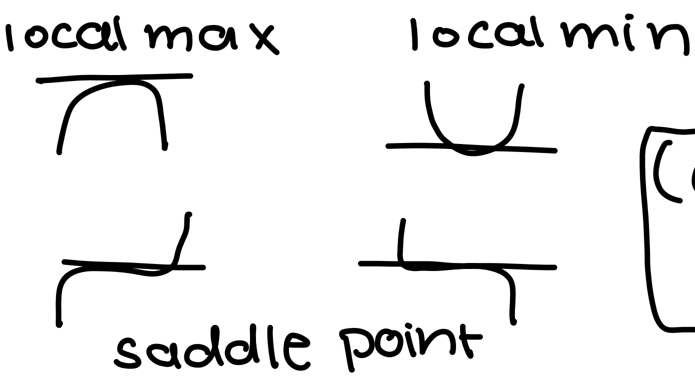
There are three flavors of such points…

Types of critical points

1. A **local max** occurs at (a,b) if f(a,b) is larger than *all* values "near" it (top of a hill).

2. A **local min** occurs at (a,b) if f(a,b) is smaller than *all* values "near" it (bottom of a valley).

3. If the partial equal zero and it is not a max or min, then we call it a **saddle point**.



Example: ***Sp'18 – Exam 2***

Find the critical points of

$$z = f(x,y) = x^3 - x^2y + y^2 - 2y.$$

① $f_x = 3x^2 - 2xy \overset{?}{=} 0$

② $f_y = -x^2 + 2y - 2 \overset{?}{=} 0$

combine (substitution)

$\boxed{\text{Step 1}}$ Factor first!

① $x(3x - 2y) = 0$

② $-x^2 + 2y - 2 = 0$

$\boxed{\text{Step 2}}$ combine!

① $\nearrow$ $x = 0 \xrightarrow{②} 0 + 2y - 2 = 0 \to y = 1$

① $\searrow$ $3x - 2y = 0 \xrightarrow{②} -x^2 + 3x - 2 = 0$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

$3x = 2y$

$y = \frac{3}{2}x$

$x = 1 \to y = \frac{3}{2}$

$x = 2 \to y = 3$

$$\boxed{(0,1) \; (1, 3/2) \; (2,3)}$$

check points!

Second Derivative Test in 3D!

Let (a,b) be a critical point.

Find all **second** partials at (a,b) and compute

$$D = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

1. If **D > 0**, then the concavity is the same in all directions. So

   (a) If **f_{xx} > 0**, then it is concave up in all directions. **Local Minimum.**

   (b) If **f_{xx} < 0**, then it is concave down in all directions. **Local Maximum.**

2. If **D < 0**, then the concavity changes in some direction. **Saddle Point**.

3. If **D = 0**, the test is **inconclusive** . (need a contour map)

$f_{xx} = 6x - 2y$

$f_{yy} = 2$

$f_{xy} = -2x$

$f_{xx}(0,1) = -2$

$f_{yy}(0,1) = 2$

$f_{xy}(0,1) = 0$

$D = (-2)(2) - 0^2$

$\boxed{D = -4}$

$\Downarrow$

Saddle point

Quick Examples:

1. $f(x,y) = 15 - x^2 - y^2$,

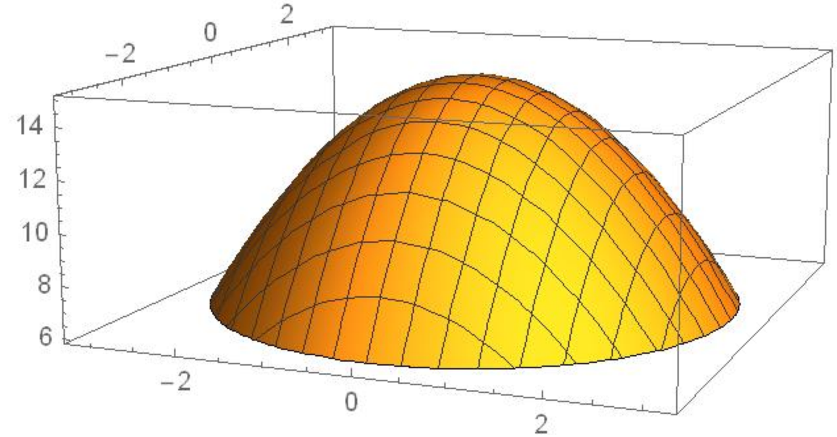
Critical pt: (0,0).

$f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 0$

$D = (-2)(-2)-(0)^2 = 4$

$D > 0$, $f_{xx} < 0$, $f_{yy} < 0$

Local max!



2. $f(x,y) = x^2 + y^2$,

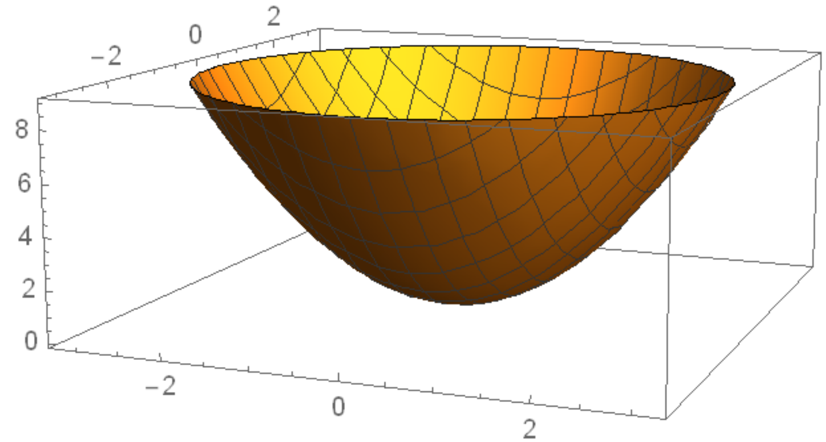
Critical pt: (0,0).

$f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$,

$D = (2)(2)-(0)^2 = 4$

$D > 0$, $f_{xx} > 0$, $f_{yy} > 0$

Local min!



3. $f(x,y) = x^2 - y^2$

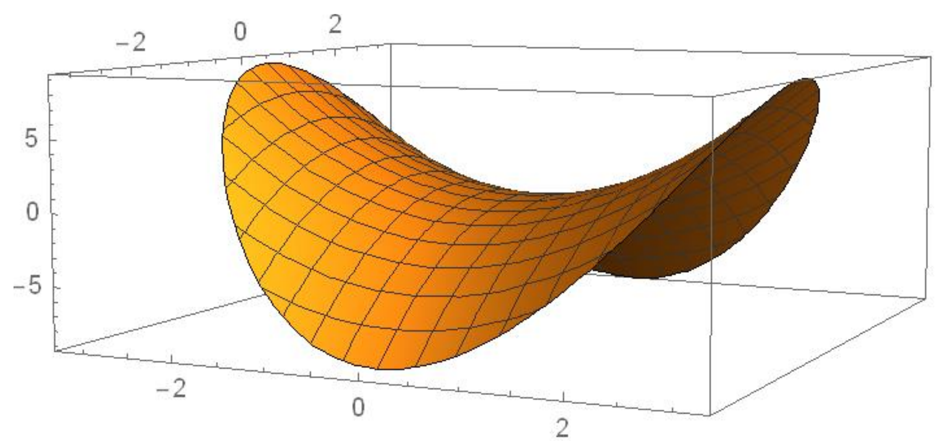
Critical pt: (0,0).

$f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = 0$,

$D = (2)(-2)-(0)^2 = -4$

$D < 0$ (note also, $f_{xx} < 0$, $f_{yy} > 0$)

Saddle point!



Sp'18 – Exam 2

Find and classify the critical points of

$$f(x,y) = x^3 - x^2y + y^2 - 2y.$$

$f_x = 3x^2 - 2xy$

$f_y = -x^2 + 2y - 2$

$3x^2 - 2xy = 0$

$x(3x - 2y) = 0$

$x = 0$ $\Downarrow$ $0 + 2y = 2$ $y = 1$ $\boxed{(0,1) \text{ saddle point}}$

$3x - 2y = 0$

$2y = 3x$

$-x^2 + 2y - 2 = 0$

$-x^2 + 3x - 2 = 0$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2)$

$x = 1$ $\Downarrow$ $2y = 3$ $y = 3/2$ $\boxed{(1, \tfrac{3}{2}) \text{ Local min}}$

$x = 2$ $\Downarrow$ $2y = 6$ $y = 3$ $\boxed{(2,3) \text{ saddle point}}$

$f_{xx} = 6x - 2y$ $\quad f_{yy} = 2$ $\quad f_{xy} = -2x$

$D(0,1) = (-2)(2) - (0) = -4$ $\quad D < 0$

$D(1, \tfrac{3}{2}) = (3)(2) - (4) = 4$ $\quad D > 0,\ f_{xx} > 0$

$D(2,3) = (6)(2) - (16) = -4$ $\quad D < 0$

Visuals: https://www.math3d.org/ySr7jcXQ

Examples from old exams:

1. Find and classify all critical points for

$$f(x,y) = x^2 + 4y - x^2y + 1$$

$f_x = 2x - 2xy \quad f_y = 4 - x^2$

$2x - 2xy = 0 \qquad 4 - x^2 = 0$

$2x(1-y) = 0 \qquad x^2 = 4$

$x = 0 \quad y = 1 \qquad x = \pm 2$

$4 - 0 = 0 \quad \cancel{4 - x^2 = 0} \quad 4 - 4y = 0$

Not possible!

$y = 1 \quad (2,1)$

$f_{xx} = 2 - 2y \qquad -4 + 4y = 0$

$f_{yy} = 0 \qquad y = 1 \quad (-2,1)$

$f_{xy} = -2x$

$\boxed{\begin{array}{l}(2,1) \Rightarrow \text{saddle point} \\ (-2,1) \Rightarrow \text{saddle point}\end{array}}$

$D(2,1) = (0)(0) - (16) = -1 \quad D<0$

$D(-2,1) = (0)(0) - (16) = -1 \quad D<0$

2. Find and classify all critical points for

$$f(x,y) = \frac{9}{x} + 3xy - y^2$$

Homework question

$f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$

$f_x = 3x^2 - 6x - 9$

$3x^2 - 6x - 9 = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3 \quad x = -1$

$f_y = 3y^2 - 6y$

$3y^2 - 6y = 0$

$3y(y-2) = 0$

$y = 0 \quad y = 2$

① $(x-3)(x+1) = 0$

$x = 3 \quad x = -1$

① $(x-3)(x+1) = 0$

$x = 3 \quad x = -1$

$$\boxed{\begin{array}{c}(3,0)\\(3,2)\\(-1,0)\\(-1,2)\end{array}} \rightarrow \text{classify!}$$

(x, y, f)

$f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$

value = D value

3. Find and classify all critical points for

$$f(x, y) = x^2y - 9y - xy^2 + y^3$$

Applied Max/Min optimization!

(a) Label Everything.

(b) ***Objective***:
What you are optimizing?!?

(c) ***Constraint***: What is given?
Use the constraints to give a two-variable function for the objective.

(d) Differentiate objective.
Find critical pts.

I looked through homework and old exams and here are the key words in all those applied optimization problems. Do you know what these words tell you to do?

"… minimum distance to a given point…."
"… closest pt. on a surface to a given pt…"

"… minimum surface area..."
"… minimum material used…"

"… minimum cost…."

"… maximum volume..."

take the derivative of . . .

HW Examples:

1. Find the dimensions of the box with volume 1000 cm^3 that has minimum surface area.

2. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to (4,2,0).

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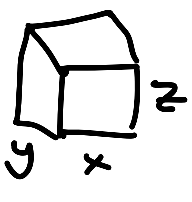
get to 2 variable function

Objective? Minimize **surface area.**

Constraint? Volume is 1000.



$$SA = 2xz + 2xy + 2yz$$

$$\text{Volume} = xyz = 1000$$

$$z = \frac{1000}{xy}$$

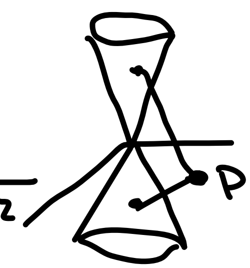
$$SA = 2x\left(\frac{1000}{xy}\right) + 2xy + 2y\left(\frac{1000}{xy}\right)$$

$$SA = \frac{2000}{y} + 2xy + \frac{2000}{x}$$

take derivatives, find critical points, solve

Objective? Min **distance** from (x,y,z) to the point (4,2,0)

Constraint? (x,y,z) must be on $z^2 = x^2 + y^2$.



$$\text{Distance} = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$$

$$\underset{(x,y)}{\text{Distance}} = \sqrt{(x-4)^2 + (y-2)^2 + x^2 + y^2}$$

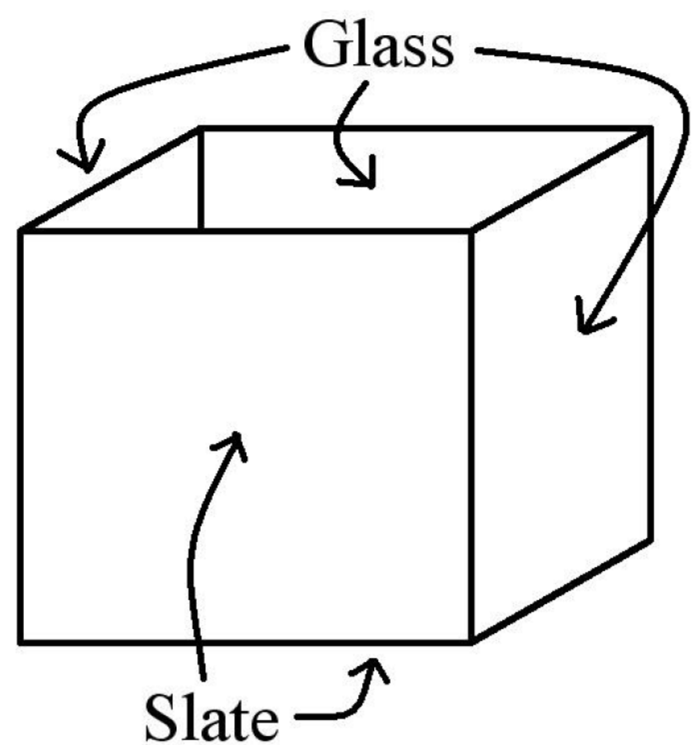
$$f_x = \frac{1}{2\sqrt{\text{mess}}}\left(2(x-4) + 2x\right) = 0$$

$$f_y = \frac{1}{2\sqrt{\text{mess}}}\left(2(y-2) + 2y\right) = 0$$

F'13 – Exam 2 – Loveless

(13 pts) You are designing a box that does not have a top (as shown).
The volume must be 10 cubic feet.
The bottom and front are are both going to be made out of slate that costs \$5.00 per square foot.
The other three sides are made of glass that costs \$2.00 per square foot.
Find the minimum cost.

Verify that your critical point is a local minimum by using the second derivative test.
(To speed up your work, you do not have to give exact answers here. Instead you should give decimal values correct to two digits after the decimal).



4. (12 pts) Find the x, y, z dimensions of the rectangular box with maximum volume in the first octant with all vertices (corners) in the coordinate planes except one vertex (corner) that is on the plane $4x + 3y + z = 12$. (One example of such a rectangular box is shown)

No points for guessing, you have to appropriately set-up a function and solve for a critical point. At the end, ***clearly*** use the 2nd derivative test to verify your point gives a local max.

volume $= xyz \Leftarrow$ optimizing volume

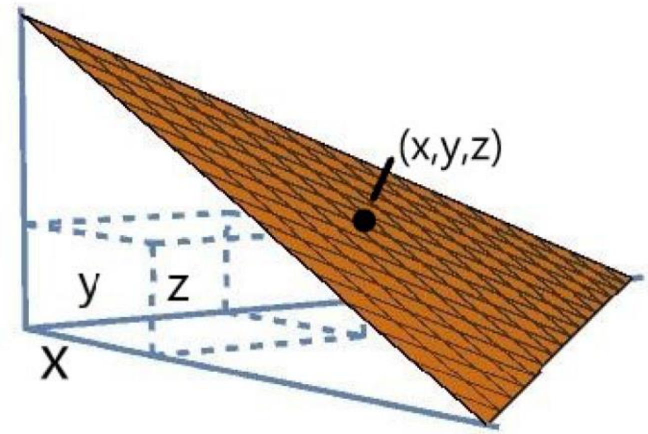
$4x+3y+z=12$

$z = 12-4x+3y$

$V(x,y) = xy(12-4x+3y)$

$V(x,y) = 12xy - 4x^2y + 3xy^2$

take derivatives + solve !



Visual: https://www.math3d.org/tgjTPIT4

Global Max/Min: Consider a surface z=f(x,y) over region R on the xy-plane. The **absolute max/min** over R are the largest/smallest z-values.

Key fact (Extreme value theorem)
The absolute max/min must occur at

1. A critical point, or
2. A boundary point.

How to do global max/min problems:
Step 1: Find critical pts inside region.
Step 2: Find critical numbers and corners above each boundary.
Step 3: Evaluate the function at all pts from steps 1 and 2.

Biggest output = global max
Smallest output = global min

Easy Example: Consider the paraboloid
$$z = x^2 + y^2 + 3$$
above the circular disk $x^2 + y^2 \leq 4$.
Find the absolute max and min.

Boundaries (step 2) details:

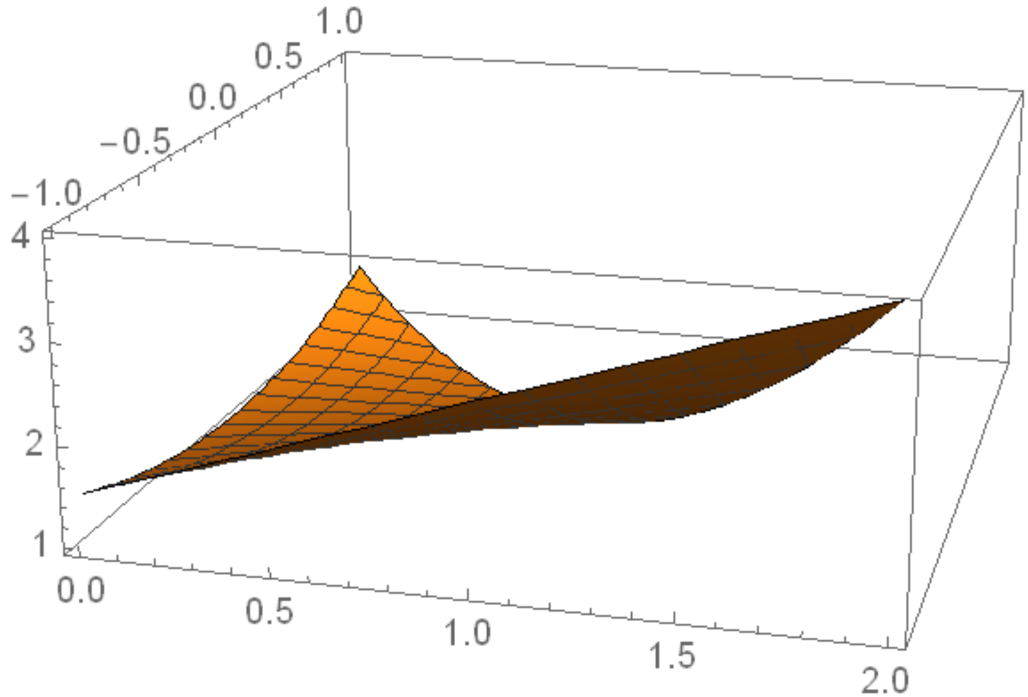
i) For each boundary, give an equation in terms of x and y. Find intersection with surface.

ii) Find critical numbers and endpoints for this one variable function. Label “corners/endpoints”.

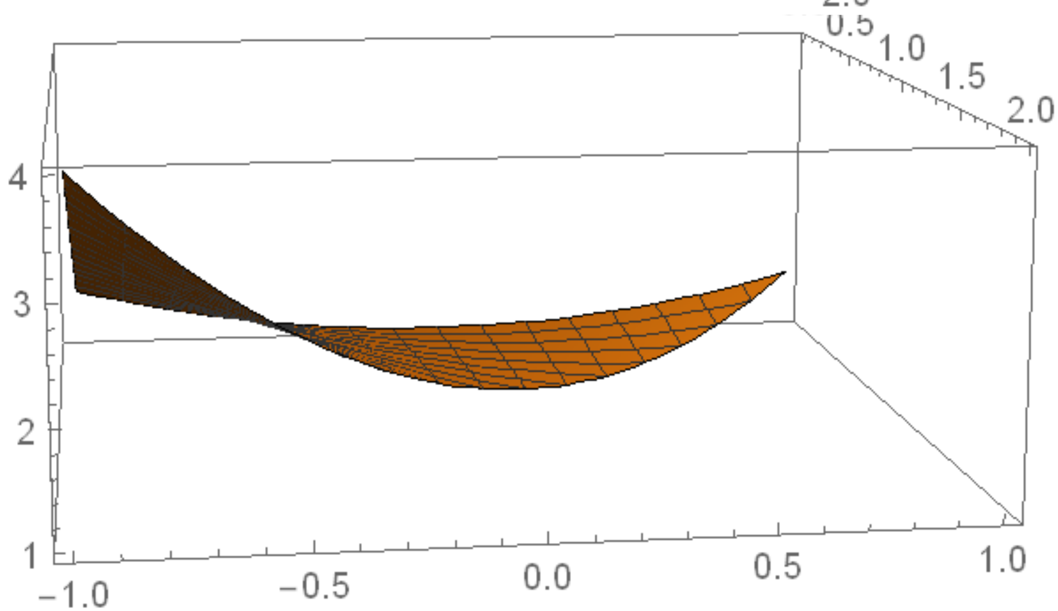
Typical Example:

Let R be the triangular region in the xy-plane with corners at (0,-1), (0,1), and (2,-1). Above R, find the *absolute* (*global*) max and min of

$$f(x,y) = \frac{1}{4}x + \frac{1}{2}y^2 - xy + 1$$



1.0
0.5
0.0
−0.5
−1.0
4
3
2
1
0.0
0.5
1.0
1.5
2.0



0.5
1.0
1.5
2.0
4
3
2
1
−1.0
−0.5
0.0
0.5
1.0

Another Example:

Find the absolute max/min of

$$f(x, y) = x^3 - 12x + y^2$$

over the region

$$x \geq 0, x^2 + y^2 \leq 9.$$